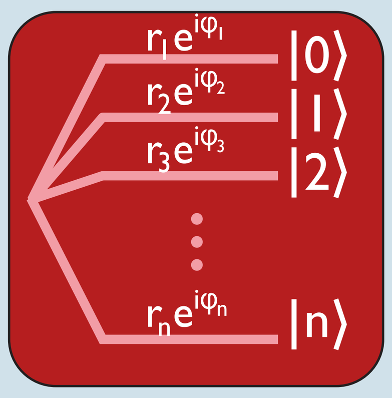
Phase and particle exchange beyond quantum theory

Andrew J. P. Garner[1] Oscar C. O. Dahlsten[1,2] Vlatko Vedral[1,2]
Yoshifumi Nakata[3] Mio Murao[3] Jayne Thompson[2] Mile Gu[2]

1. Introduction



In quantum theory different **phases** φ_i can be associated with different branches of a superposition by polar decomposing the complex amplitudes of each branch: $|\psi\rangle = \sum_j r_j \exp(i\varphi_j)|j\rangle$.

Whilst these phases are not observable in the basis in question, $\{|j\rangle\}$, they can have significant consequences for measurements in other bases. Phase plays a fundamental role in many of the most strikingly non-classical behaviours of quantum theory, in particular those relating to interference, such as phase estimation (the basis of most quantum algorithms) and particle exchange statistics. Understanding the phenomenon of phase should be a key aim of research efforts into quantum foundations.

It is not a priori well-defined to talk about phase in experiments involving systems not governed by quantum theory as the definition of phase is tied into the formalism of quantum theory. However, there is currently great interest in parts of the quantum information and quantum foundations communities in investigating theories in a wider framework than quantum theory known as **generalised probabilistic theories** (GPTs), also called the **convex framework**. A key motivation is to better understand quantum theory, but one may also consider that something beyond quantum theory may actually be realized in some circumstances.

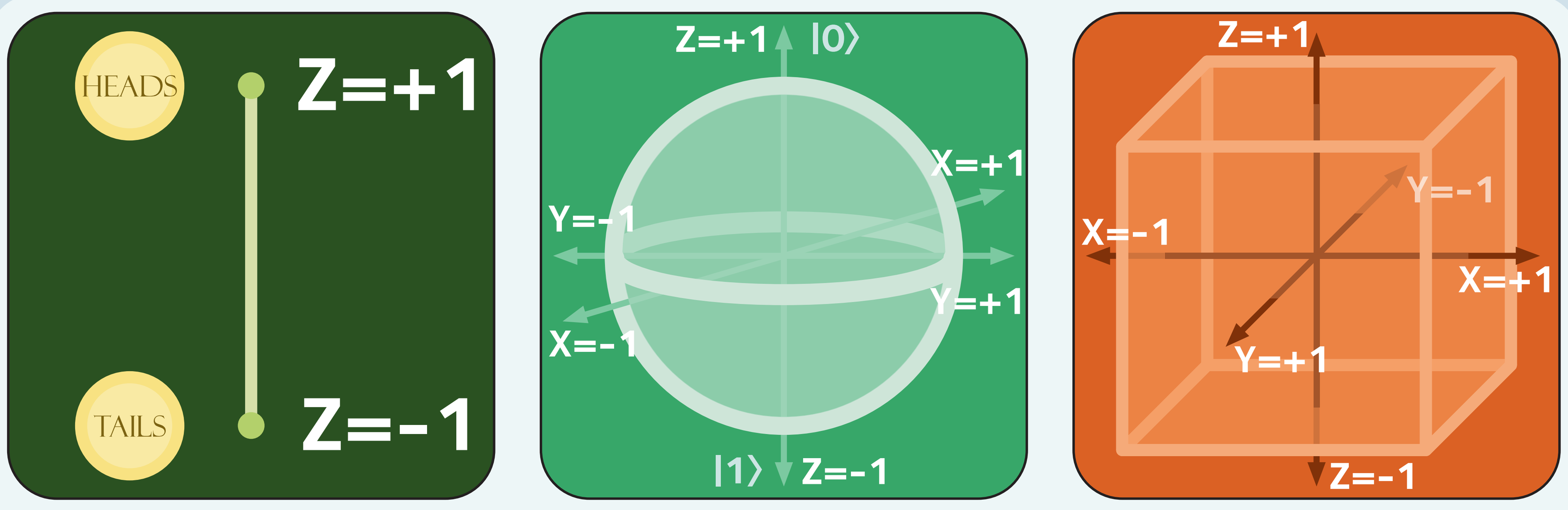
Here we aim to lay the foundations for studying **phase** in GPTs. We focus on how to define phase, and on the role it plays in **particle exchange statistics** in GPTs.

2. The GPT framework

States

In the **Generalised Probabilistic Theory** (**GPT**) framework, it possible to operationally describe the state of a physical system by listing the complete set of probabilities of the different outcomes of the next measurement made on the system. In a classical system, the state is uniquely specified by the probabilities associated with a single measurement, but in general there may be more than one measurement.

States may be written as a vector of real numbers. The set of all allowed states in a theory combine to form a **state space**, as shown in the examples below:



Classical

The simplest classical state space is that of a **bit**.

The two end points of this line correspond to when a flipped coin is known to definitely be in a heads state (or a tails state); the connecting line shows states of varying uncertainty (the midpoint corresponds to total ignorance of the coin's state).

Quantum

In this framework, the set of states a single **qubit** can take is represented by the Bloch sphere. There are three orthogonal binary measurements which could be made on the system.

Pure states, which can not be formed from mixing other states, lie on the surface. The **mixed states** are within.

Beyond Quantum

One GPT beyond quantum theory is the **gbit** (generalised bit): the extreme case in which every statistically possible state is allowed. The corresponding state space for binary measurements is a (hyper-)cube.

This includes states not allowed by quantum mechanics, such as the corner states, where the outcome of every measurement can be perfectly predicted.

Effects

It is possible to associate an **effect** with every outcome of a measurement. The inner product of this effect with a state gives the probability of the associated outcome if one made that measurement on the state. Within a theory a measurement can be characterised by its set of effects.

Transformations

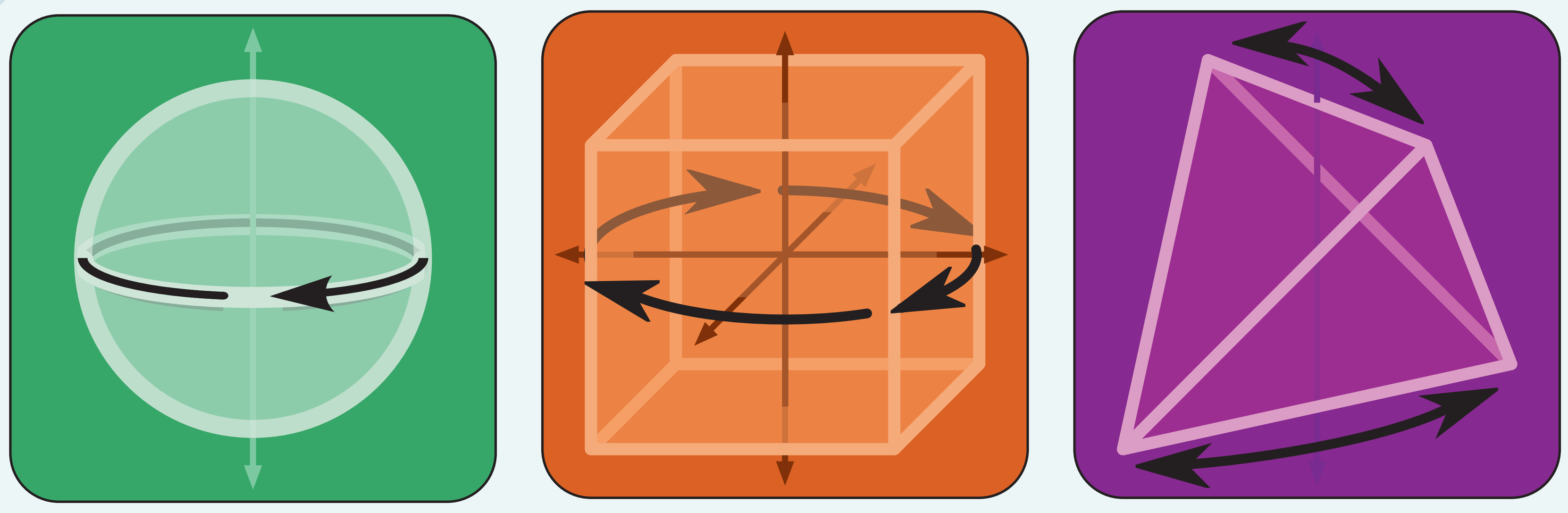
A **transformation** is a map that takes any allowed state in a theory to another allowed state. If the state space before transformation is the same as the set of states after, then the transformation is **reversible**. Only imposing the restriction of linearity, this means that the maximal set of allowable reversible transformations are in fact the automorphism group of the state space (i.e. its symmetries).

3. Phase groups

For a given quantum basis, there is a set of unitary operations which do not alter the outcomes of measurement in this basis: $U = \sum_j \exp(i\varphi_j)\, |j\rangle\langle j|$. We generalise this into the GPT framework by taking the set of reversible transformations T which do not alter the measurement statistics of a measurement, $\{e_i\}$, for all states s, satisfying:

$$e_i \cdot s = e_i \cdot (Ts)$$

This condition induces a subgroup from the total set of reversible transformations, which is referred to as the **phase group** associated with the measurement. A few examples of phase groups associated with binary measurement are shown below:



Qubit *SO(2)*

3-in 2-out gbit *Symmetries of a square*

Tetrahedral bit $Z_2 \oplus Z_2$

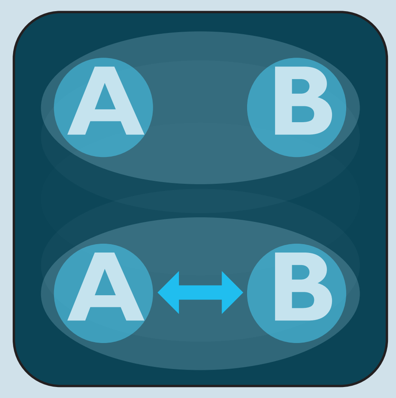
The phase group of a maximally fine-grained measurement on a classical system is always just the set containing the identity: the **trivial phase group**.

For an interferometer described by quantum theory, it is possible to apply an element of the phase group through a local action, such as choosing to add a piece of glass in one branch.

However, for some theories (including some gbits) the phase group may be **non-abelian**, and it is dangerous to trivially associate elements of this group with local operations. By considering relativity of simultaneity, operations occurring on spatially-separated branches may be observed as occurring in different orders by different observers, and as these operations might not commute, the output statistics which should be Lorentz invariant (measured by cascades in particles detectors) will differ.

4. Particle exchange statistics

In quantum theory, under exchange particles may behave like bosons, fermions or more generally, anyons. For two particles A and B in a pure state $|AB\rangle$ and a swap operation Π, if A and B are indistinguishable, we require $\Pi|AB\rangle = e^{i\theta}|AB\rangle$; the input and output differ only by a global phase determined by the type of particle. To observe this phase we may introduce an ancillary control system C, such that the swap is only performed if C is in a particular state (e.g. Z=+1, $|1\rangle$).



We can physically envision this set-up as two traps each containing one particle. The traps are split into a spatially disjoint superposition such that either both particles are on the top, or both are on the bottom. The traps on the bottom are swapped around each other, whilst those on the top are unchanged.

If the control system is initially in a superposition state $|0\rangle+|1\rangle$, the controlled-swap evolves the system as:

$$(|0\rangle + |1\rangle)|AB\rangle \longrightarrow (|0\rangle + e^{i\theta}|1\rangle)|AB\rangle,$$

a phase transform with respect to the Z basis of the control bit.

Extension to the GPT framework

We now consider this process in the GPT framework. We take as the definition of indistinguishable that swapping A and B leaves the combined state S_{AB} invariant (even if embedded in a larger system). If we consider a product state of AB and C, and applying a reversible swap operation, we find:

$$S_{AB}^{in} \otimes S_C^{in} \longrightarrow S_{AB}^{out} \otimes S_C^{out} = S_{AB}^{in} \otimes S_C^{out}.$$

From this, we see that the only observable effect of the exchange is on the control bit C. Furthermore as the control bit could refer to disjoint branches in space, in order to avoid signalling the transformation must be within the phase group of C.

The observable effect of exchanging two indistinguishable particles is restricted to the control system being transformed by an element of the phase group of a binary measurement. The particle types perceivable in a theory correspond to elements of this phase group.

1. Atomic and Laser Physics, Clarendon Laboratory, University of Oxford, Parks Road, Oxford, OX1 3PU, United Kingdom
2. Center for Quantum Technologies, National University of Singapore, Republic of Singapore
3. Department of Physics, University of Tokyo, Tokyo 113-0033, Japan

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